The Effect of Direct Current Bias in the Computation of Power Spectra

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We determine the effect of dc bias in the approximate computation of spectra of Gaussian processes by hard limiting.

I. Introduction

The spectral density $S(\omega), |\omega| \leq \pi$, of a stationary Gaussian process $\{x_k\}$, $-\infty < k < \infty$, of mean zero, can be expressed in terms of the covariances $R_x(k) = E(x_n x_{n+k})$ by

$$S(\omega) = R_x(0) + 2\sum_{k=1}^{\infty} R_x(k) \cos k\omega. \tag{1}$$

If we put

$$y_i = \begin{cases} +1, x_i \geq 0, \\ -1, x_i < 0, \end{cases} \tag{2}$$

Then it is known (Ref. 1) that $R_y(k) = E(y_n y_{n+k})$ satisfies the relation

$$R_x(k) = R_x(0) \sin\left[\frac{\pi}{2} R_y(k)\right]. \tag{3}$$

In practice, the series in (1) is truncated, and the correlations $R_y(k)$ are estimated from samples of finite size, which leads to random errors in the evaluation of $S(\omega)$. These errors will not be considered here. Neglecting them, we use the variables $\{y_k\}$ to estimate $R_y(k)$, then apply (3) and (1) to get $S(\omega)$. This leads to a saving in computation time over the direct estimation of $R_x(k)$ from $\{x_k\}$, since these estimates require a large number of arithmetic operations. The variance $\sigma^2 = R_x(0) = E(x_k^2)$, if needed, must be estimated separately. This variance, which enters $S(\omega)$ as a scale factor, is often unimportant, because it can be affected by many extraneous factors.

We consider the following problem: Suppose that an unknown bias a is added to the Gaussian process $\{x_k\}$, so that we get $\{x_k + a\}, -\infty < k < \infty$. The formula (2) cannot be applied if a is unknown. However, we can take

$$y_i = \begin{cases} +1, x_i + a \geq 0, \\ -1, x_i + a < 0. \end{cases} \tag{4}$$

Then we can estimate the numbers $E_k = E(y_n y_{n+k})$ (no longer correlations) take

$$R_x^*(k) = \sin\left(\frac{\pi}{2} E_k\right), \tag{5}$$

in analogy with (3), and form

$$S^*(\omega) = R_x^*(0) + 2\sum_{k=1}^{\infty} R_x^*(k) \cos k\omega. \tag{6}$$

If $a = 0$, $S^*(\omega)$ is $S(\omega)/\sigma^2$. If $a \neq 0$, $S^*(\omega)$ approximates $S(\omega)$ in some sense. We shall investigate how close this approximation is.

To find $S(\omega)$ exactly, (3) needs to be replaced by a formula involving a non-elementary function which depends on a/σ. This is not practical for real-time computations. Hence we consider the above approximation.

II. Formulas

Let the Gaussian random variables x_k have variance σ^2. Then the value of $E_k = E(y_n y_{n+k})$ depends only on $\rho_k = R_x(k)/\sigma^2$, and can be written as

$$E_k = f(\rho_k), \tag{7}$$

where

$$f(z) = \frac{1}{2\pi\sqrt{1-z^2}} \times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y(t_1)y(t_2) \exp\left[\frac{t_1^2 + t_2^2 - 2zt_1t_2}{2(1-z^2)}\right] dt_1\, dt_2, \tag{8}$$

with

$$y(t) = \begin{cases} +1, t \geq -a/\sigma, \\ -1, t < -a/\sigma. \end{cases}$$

This follows if we evaluate E_k directly as the expectation of a function of x_n and x_{n+k}, by integrating over the bivariate Gaussian distribution (Ref. 2).

If we differentiate (8) with respect to z, the resulting integral can be simplified by integration by parts, and we get

$$f'(z) = \frac{2}{\pi\sqrt{1-z^2}} \exp\left(-\frac{a^2/\sigma^2}{1+z}\right).$$

Integrate this equation from 0 to z. Using the variable of integration $u = \sin^{-1}z$,

$$f(z) - f(0) = \frac{2}{\pi}\int_0^{\sin^{-1}z} \exp\left(-\frac{a^2/\sigma^2}{1+\sin u}\right) du. \tag{9}$$

To evaluate $f(0)$, note that at $z = 1$, the distribution in (8) is concentrated on the line $t_1 = t_2$. Then $f(1) = E(y(t_1)^2) = 1$. Hence, putting $z = 1$ in (9),

$$1 - f(0) = \frac{2}{\pi}\int_0^{\pi/2} \exp\left(-\frac{a^2/\sigma^2}{1+\sin u}\right) du. \tag{10}$$

By numerical integration of the integral in (9), going from $z = 0$ to $z = 1$ and to $z = -1$, tables of $f(z)$ were obtained for various values of a/σ. Some of these are plotted in Fig. 1.

If we plot $[\sin(\pi/2)f(z)]$ instead of $f(z)$, the curve for $a = 0$ becomes a straight line, and the other curves are also modified, but they do not come significantly closer to the line for $a = 0$. The separation between the curves is the error in this method of approximating $R_x(k)/\sigma^2$, so at first glance this method does not look too promising.

However, instead of $R_x^*(k)$, we should be considering

$$\overline{R_x}(k) = \frac{\sin\left[\frac{\pi}{2}E_k\right] - \sin\left[\frac{\pi}{2}f(0)\right]}{\sin\left[\frac{\pi}{2}f(1)\right] - \sin\left[\frac{\pi}{2}f(0)\right]} = C_1\,[R_x^*(k) - C_2], \tag{11}$$

where

$$C_2 = \sin\left[\frac{\pi}{2} f(0)\right],$$

and

$$C_1 = (1 - C_2)^{-1}.$$

$\overline{R_x}(k)$ is plotted as a function of $R_x(k)/\sigma^2$ in Fig. 2. We see that $\overline{R_x}(k)$ is a good approximation to $R_x(k)/\sigma^2$ if $R_x(k)/\sigma^2$ is not too close to -1 (which is usually true).

Forming the numbers $\overline{R_x}(k)$ requires knowing something about the function $f(z)$. However, they are only wanted to construct the function

$$\overline{S}(\omega) = 1 + 2 \sum_{k=1}^{\infty} \overline{R_x}(k) \cos k\omega. \tag{12}$$

This function can be found more directly as

$$\overline{S}(\omega) = C_1[S^*(\omega) - C_2\delta(\omega)]. \tag{13}$$

The presence of a dc component in $\{y_k\}$ causes a spike in the spectrum $S^*(\omega)$ at $\omega = 0$. This can be removed by inspection, which is the significance of the term $-C_2\delta(\omega)$ in (13). Then the multiplier C_1 can be adjusted to give whatever scale is desired for the spectrum. Thus we get $\overline{S}(\omega)$, defined by (12), without any particular knowledge about a or the function f(z).

The error $\overline{R_x}(k) - R_x(k)/\sigma^2$ is negative for $R_x(k) > 0$, with a minimum near $R_x(k) = 0.44\sigma^2$ for all values of a/σ between 0 and 1. For $R_x(k) < 0$, the error is positive, increasing as $R_x(k)/\sigma^2$ goes from 0 to -1. These errors are given at $R_x(k)/\sigma^2 = 0.44$ and -0.30 for several values of a/σ in Table 1.

Another method which we can consider for comparison is to first compute the correlation

$$R_y(k) = \frac{E_k - f(0)}{1 - f(0)},$$

then take

$$\widetilde{R}_x(k) = \sin\left[\frac{\pi}{2} R_y(k)\right]$$

and

$$\widetilde{S}(\omega) = 1 + 2 \sum_{k=1}^{\infty} \widetilde{R}_x(k) \cos k\omega.$$

This method is suggested by the original relation (3). Curves for $\widetilde{R}_x(k)$ as a function of $R_x(k)/\sigma^2$, similar to the curves of Fig. 2, can be plotted. These curves have the same general appearance, but they range farther from the line for $a = 0$. Values of the error $\widetilde{R}_x(k) - R_x(k)/\sigma^2$ are given in Table 1 for comparison. We see that this method is not as good, as well as being harder to implement.

References

1. Goldstein, R. M., "Radar Exploration of Venus," Thesis, California Institute of Technology, 1962.

2. Feller, W., *An Introduction to Probability Theory and Its Applications*, Vol. 2, 2nd edition, Wiley, New York, 1961.

Table 1. The errors in $R_x(k)$ and $R_x(k)$ for $R_x(k)/\sigma^2 = 0.44$ (upper number) and -0.30 (lower number)

a/σ	0.2	0.5	1.0
$\overline{R_x}(k) - R_x(k)/\sigma^2$	-0.0001 $+0.0003$	-0.0031 $+0.0093$	-0.0342 $+0.0822$
$\widetilde{R}_x(k) - R_x(k)/\sigma^2$	-0.0031 $+0.0064$	-0.0192 $+0.0378$	-0.0738 $+0.1261$

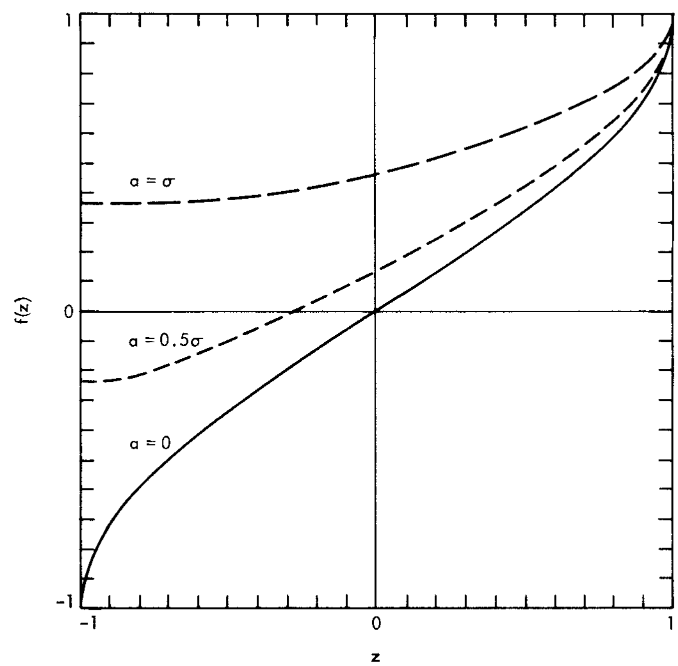

Fig. 1. The function *f(z)* for various values of a/σ

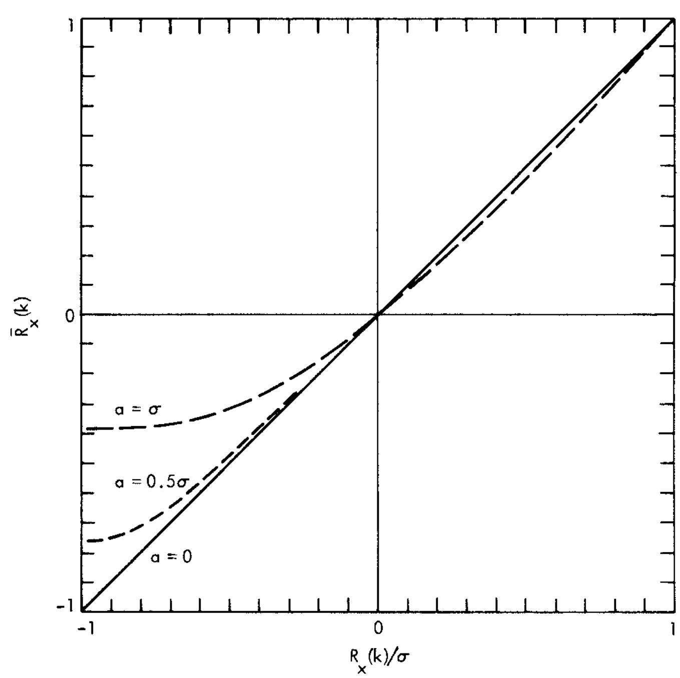

Fig. 2. $\overline{R_x}(k)$ as a function of $R_x(k)/\sigma^2$, for various values of a/σ